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Block Spin Approach to ϕ_3^4 Field Theory

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A block spin approach to the Euclidean ϕ^4 field theory in three dimensions is proposed by using the three-dimensional version of Gawedzki and Kupiainen's block spin transformation method. The lattice ϕ_3^4 model recovers the rotation invariance in the continuum limit, when the coupling constant is small.

KEY WORDS: ϕ_3^4 field theory; lattice regularization; continuum limit; rotation invariance; block spin transformation method.

1. INTRODUCTION

Euclidean ϕ_3^4 field theory[1] has been developed by many authors and has become a mature branch of mathematical physics (see the list of papers in ref. 2). Among others, Brydges *et al.*[2] invented a simple and sophisticated construction: they found a clear picture obtaining a continuum limit without any renormalization group insight. However, their program to derive the rotation invariance and the uniqueness of the continuum limit seems somewhat complicated, and no one has succeeded in completing it. At the least, we have learned how far one can go without the renormalization group philosophy and realize where it is needed.

On the other hand, the block spin transformation method, which is a mathematical realization of the renormalization group philosophy, yielded detailed information on the ϕ_4^4 model.[3-5] While these works are concerned with problems on triviality, their technique is not restricted to the direction of triviality but is applicable to a model with a small coupling constant. Therefore (even if the critical phenomena in three dimensions are beyond our scope) it will be possible to study the ϕ_3^4 field theory by the block spin transformation method, since its lattice approximation has a small coupling constant.

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In this paper, I propose the block spin approach to the ϕ_3^4 field theory and confirm that the lattice approximation considered in ref. 2 recovers the rotation invariance[6] in (each) continuum limit if the coupling constant is small. The method employed here is the three-dimensional version of the block spin analyses in refs. 3–5: I trace the trajectories of the block spin transformation for lattice ϕ_3^4 models and estimate the mutual difference between two near trajectories. Applying the results to the "rotated" system, I derive the rotation invariance as an asymptotic disappearance of an irrelevant perturbation.

A similar picture was pursued in ref. 8, which studied the $U(1)$ Higgs model in three dimensions. However, we have to look into *all* the Schwinger functions and derive their rotation invariance, while in ref. 8, as long as the ϕ^4 part is concerned, only a certain class of Schwinger functions is studied. See also ref. 7.

I proceed as follows. In Section 2, I review the lattice approximation studied in ref. 2 and state my result together with a rough sketch of the method. In Section 3, I describe the outline of the block spin analysis of the lattice ϕ_3^4 model and the scheme estimating the free energy of the effective theory. By applying the result in Section 3 to the lattice approximation, I show the rotation invariance in Section 4. The properties assumed for the "rotated" Gaussian kernel are shown in the Appendix.

2. THE MODEL AND THE RESULT

Here I introduce the lattice approximation for the ϕ_3^4 field theory argued in ref. 2. Put $L = 5^M$ for a sufficiently large integer M. Let N_1 and N_2 be positive integers and put $N = N_1 + N_2$. Now, discretizing $\mathbb{R}^3$ as $L^{-N_1}\mathbb{Z}^3$ and periodizing $\mathbb{Z}^3$ as $\Lambda_0 = (\mathbb{Z}/L^N\mathbb{Z})^3$, define the regularized Schwinger functions by

$$S_{2l}^{(N_1,N_2)}(x_1, x_2,..., x_{2l}) = \int d\mu^{(N_1,N_2)}(\phi)\, \phi_{L^{N_1}x_1} \phi_{L^{N_1}x_2} \cdots \phi_{L^{N_1}x_{2l}}$$
$$x_1, x_2,..., x_{2l} \in L^{-N_1}\Lambda_0 \tag{2.1}$$

where $d\mu^{(N_1,N_2)}(\phi)$ is the lattice ϕ^4 measure on Λ_0 given by

$$d\mu^{(N_1,N_2)}(\phi) = \exp[-A^{(N_1,N_2)}(\phi)] \prod_{u\in\Lambda_0} d\phi_u/\text{normalization} \tag{2.2}$$

$$A^{(N_1,N_2)}(\phi) = \frac{1}{2} L^{-N_1} \sum_{\substack{u,v\in\Lambda_0\\ \text{n.n.}}} |\phi_u - \phi_v|^2 + \frac{1}{2} L^{-3N_1}(m_*^2 - c) \sum_{u\in\Lambda_0} \phi_u^2 + L^{-3N_1}\lambda_* \sum_{u\in\Lambda_0} \phi_u^4 \tag{2.3}$$

I choose the constant c later [see (2.7)]. Replace the variable ϕ_u by $L^{N_1/2}\phi_u$ and rewrite the right-hand side of (2.1) as

$$S_{2l}^{(N_1,N_2)}(x_1, x_2,..., x_{2l}) = L^{N_1 l} \int d\mu_{G^{\rm st}}(\phi) \exp[-V^{(0)}(\phi)]\, \phi_{L^{N_1}x_1}\phi_{L^{N_1}x_2}\cdots\phi_{L^{N_1}x_{2l}} \qquad (2.4)$$

Here, $d\mu_{G^{\rm st}}(\phi)$ denotes the standard massive Gaussian measure:

$$d\mu_{G^{\rm st}}(\phi) = \exp\left(-\frac{1}{2}\sum_{\substack{u,v\in\Lambda_0\\ \text{n.n.}}} |\phi_u-\phi_v|^2 - \frac{1}{2}m_0^2 \sum_{u\in\Lambda_0}\phi_u^2\right) \times \prod_{u\in\Lambda_0} d\phi_u/\text{normalization} \qquad (2.5)$$

with the covariance $G^{\rm st}$, and the potential $V^{(0)}(\phi)$ is given by

$$V^{(0)}(\phi) = \lambda_0 \sum_{u\in\Lambda_0}\phi_u^4 - 6\lambda_0 \sum_{u\in\Lambda_0} G_{uu}^{\rm st}\phi_u^2 + 48\lambda_0^2 \sum_{u\in\Lambda_0}\sum_{v\in\Lambda_0}(G_{uv}^{\rm st})^3\phi_u^2 \qquad (2.6)$$

where I have put $m_0 = L^{-N_1}m_*$, $\lambda_0 = L^{-N_1}\lambda_*$, and have chosen the constant c so that

$$-\frac{1}{2}L^{-2N_1}c = -6\lambda_0 G_{uu}^{\rm st} + 48\lambda_0^2 \sum_{v\in\Lambda_0}(G_{uv}^{\rm st})^3 \qquad (2.7)$$

holds. The above choice is essentially the same as the one in ref. 2.

Now take the infinite-volume limit:

$$S_{2l}^{(N_1)}(x_1, x_2,..., x_{2l}) = \lim_{N_2\to\infty} S_{2l}^{(N_1,N_2)}(x_1, x_2,..., x_{2l}), \qquad x_1, x_2,..., x_{2l} \in L^{-N_1}\mathbb{Z}^3 \qquad (2.8)$$

I shall show later (Section 3.2, Remark 2) that the lattice system lies in the high-temperature phase for any N_1 if $\lambda_* > 0$ is sufficiently small for a fixed $m_* > 0$. Then the infinite-volume limit is independent of the boundary conditions, and hence there is no difference between the present $S_{2l}^{(N_1)}$ and the one in ref. 2.

Next consider the continuum limit $N_1 \to \infty$. Assume that $\lambda_* > 0$ is sufficiently small for a fixed $m_* > 0$. It was shown[2] that, for each l, the series $(S_{2l}^{(N_1)})_{N_1=1,2,...}$ is bounded in the space $\mathscr{S}'(\mathbb{R}^{6l})$ of the tempered distributions on $\mathbb{R}^{6l}$, and hence that there is a convergent subsequence

$(S_{2l}^{(N_1^{(j)})})_{j=1,2,\ldots}$. (The result in ref. 2 is in fact stronger than the above: the lattice spacing is not restricted to L^{-N_1}, $N_1 = 1, 2, \ldots$.) Now put

$$S_{2l} = \lim_{j \to \infty} S_{2l}^{(N_1^{(j)})} \tag{2.9}$$

Theorem. The distribution S_{2l} has the rotation invariance if $\lambda_* > 0$ is sufficiently small for a fixed $m_* > 0$.

The proof is based on the three-dimensional version of Gawedzki and Kupiainen's block spin transformation method and its extension by Hara and Tasaki. The program of the proof is as follows.

1. The problem is reduced to a comparison between the lattice system and the "rotated" system by defining properly the rotation of the lattice.

2. The difference between the original and the rotated lattice systems turns out to be a perturbation of the Gaussian measure. Then, based on the analysis of the rotated Gaussian measure in the Appendix, one can apply the bock spin analysis in Section 3. The effective potentials of the original and the rotated systems turn out to be close to each other after sufficiently many iterations of the block spin transformations.

3. In the above, the loss of information due to the coarse-graining character of the block spin transformation might seem troublesome, but this is not the case: when iterating the block spin transformation, the difference between their effective theories tends to zero uniformly in the (original) lattice spacing L^{-N_1}. Therefore, if one starts with a sufficiently fine lattice, one can perform the block spin transformations sufficiently many times, while the lattice spacing of the effective theory is arbitrarily small. This means that one can deduce the rotation invariance with arbitrary accuracy.

4. Lastly one has to evaluate the correlation function. For this purpose, a quadratic term $\sum_{u,v} h_{uv}\phi_u\phi_v$ with complex coefficients h_{uv} is added to the nth effective potential at a proper stage of the iterations where the loss of information is still small. Then the block spin transformation is continued until the effective mass becomes sufficiently large. The large mass makes it possible to estimate the free energy as a holomorphic function of h_{uv}'s (see ref. 5), and hence one can bound the correlations (and its differences), which are derivatives of the free energy with respect to h_{uv}'s. Note that λ_* must be sufficiently small for a fixed m_* since the effective coupling constant must be small when the effective mass becomes large.

3. BLOCK SPIN ANALYSIS

Let us define the ϕ^4 model on $\Lambda_0 = (\mathbb{Z}/L^N\mathbb{Z})^3$ in a slightly general context. Consider the following ϕ^4 measure on $\mathbb{R}^{\Lambda_0}$:

$$d\mu_{\lambda_0, G^{(0)}}(\phi) = e^{-V^{(0)}(\phi)}\, d\mu_{G^{(0)}}(\phi)/\text{normalization} \tag{3.1}$$

where $d\mu_{G^{(0)}}(\phi)$ is the Gaussian measure on $\mathbb{R}^{\Lambda_0}$ with mean 0 and covariance $G^{(0)}$, and the potential $V^{(0)}(\phi)$ is given by

$$\begin{aligned} V^{(0)}(\phi) = \lambda_0 \sum_{u \in \Lambda_0} \phi_u^4 - 6\lambda_0 \sum_{u \in \Lambda_0} G_{uu}^{(0)} \phi_u^2 \\ + 48\lambda_0^2 \sum_{u \in \Lambda_0} \sum_{v \in \Lambda_0} (G_{uv}^{(0)})^3 \phi_u^2 \end{aligned} \tag{3.2}$$

I do not restrict the Gaussian measure $d\mu_{G^{(0)}}(\phi)$ to the standard one with the ferromagnetic nearest neighbor interactions, but admit ones that satisfy the following properties:

(G1) $G^{(0)}$ is a positive-definite symmetric matrix with the Ornstein–Zernike decay:

$$0 \leqslant G_{uv}^{(0)} \leqslant \gamma_1 (1 + |u - v|)^{-1} \exp(-\mu_0 |u - v|) \tag{3.3}$$

(G2) $\Gamma^{(n)}$ and $\mathscr{A}^{(n)}$ defined similarly as in ref. 3 have the following estimates:

$$|\Gamma_{uv}^{(n)1/2}| \leqslant \gamma_2 e^{-2\beta |u-v|}$$

$$|\mathscr{A}_{xu}^{(n)}| \leqslant \gamma_3 e^{-2\beta |x-u|}$$

$$\left| \frac{\mathscr{A}_{xu}^{(n)} - \mathscr{A}_{yu}^{(n)}}{x - y} \right| \leqslant \gamma_4 (e^{-2\beta |x-u|} + e^{-2\beta |y-u|}), \qquad x \neq y$$

(G3) It holds that

$$\gamma_5 \mu_0^{-2} \leqslant \sum_{v \in \Lambda_0} G_{uv}^{(0)} \leqslant \gamma_6 \mu_0^{-2}, \qquad u \in \Lambda_0 \tag{3.4}$$

3.1. Trajectory

Under the above assumptions, according to the program described in ref. 3, one can explicitly determine the forms of the effective potentials within the second-order perturbation and obtain bounds on the higher-order terms and on the large-field contributions. In particular, one has to get rid of the ultraviolet divergences using the mass counterterms. Consider the following form of the quadratic part of the nth effective potential:

$$V_2^{(n)}(\Psi) = -6\lambda_n$$  $$+48\lambda_n^2 \int_{L^{-n}\Lambda_0}^{(n)} dx \int_{L^{-n}\Lambda_0}^{(n)} dy\, (L^n G^{(0)}_{L^n x, L^n y})^3\, \Psi_x^2$$

$$-48\lambda_n^2$$  $$+72\lambda_n^2$$ 

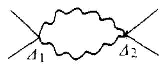
$$-72\lambda_n^2$$  $$+ \sum_{Y \subset L^{-n}\Lambda_0} \tilde{\tilde{V}}{}^{(n)}_{2,Y}(\Psi) \qquad (3.5)$$

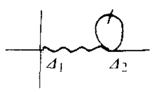
where the integral notation denotes the Riemann sum and

$$\lambda_n = L^n \lambda_0, \qquad \mathscr{G}^{(n)}_{xy} =$$ $$, \qquad \mathscr{Q}^{(n)}_{xy} =$$ 

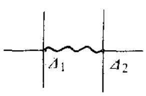
Cancellations of the ultraviolet singularities occur between the second and third terms and also *in* the first term. On the other hand, the quartic and the sextic parts are given by

$$\tilde{V}^{(n)}_{4,Y}(\Psi) = \sum_{Y=\Delta} \lambda_n$$ 

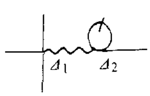
$$+ \sum_{\substack{\Delta_1 \cup \Delta_2 = Y \\ |\Delta_i| = 1}} \Bigg(-36\lambda_n^2$$ 

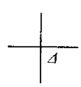
$$+48\lambda_n^2$$  $$\Bigg) + \tilde{\tilde{V}}{}^{(n)}_{4,Y}(\Psi) \qquad (3.6)$$

and

$$\tilde{V}^{(n)}_{6,Y}(\Psi) = -8\lambda_n^2 \sum_{\substack{\Delta_1 \cup \Delta_2 = Y \\ |\Delta_i| = 1}}$$  $$+ \tilde{\tilde{V}}{}^{(n)}_{6,Y}(\Psi) \qquad (3.7)$$

where

 $$= \int_{\Delta_1}^{(n)} dx \int_{\Delta_2}^{(n)} dy\, \Psi_x^3\, \mathscr{Q}^{(n)}_{xy}\, \mathscr{G}^{(n)}_{yy}\, \Psi_y,$$

 $$= \int_{\Delta}^{(n)} dx\, \Psi_x^4, \qquad \text{etc.}$$

These results together with the necessary bounds ensure that the block spin transformation can be iterated while the effective coupling constant λ_n is small. Note that I do not perform the correction of the coupling constant. For the present purpose, such a procedure is not only needless but also harmful, since one must dispense with a bound on the difference between $(\mathscr{A}_{xu}^{(n)} - \mathscr{A}_{yu}^{(n)})/|x-y|$ and $(\dot{\mathscr{A}}_{xu}^{(n)} - \dot{\mathscr{A}}_{yu}^{(n)})/|x-y|$. By the same reason, I omit the correction of the mass.

Now consider the difference of two near trajectories. For Gaussian covariances $G^{(0)}$ and $\dot{G}^{(0)}$ satisfying (G1)–(G3) (and their dotted versions), I further assume the following properties.

(ΔG1) For $x, y \in L^{-n}\Lambda_0$, it holds that

$$\begin{aligned}
&|L^n G^{(0)}_{L^n x, L^n y} - L^n \dot{G}^{(0)}_{L^n x, L^n y}| \\
&\quad \leqslant \gamma_7 L^{-n/2} \qquad\qquad\qquad \text{if} \quad x = y \\
&\quad \leqslant \gamma_7 L^{-n/2}(|x-y|^{-2}+1) \qquad \text{if} \quad x \neq y
\end{aligned}$$

(ΔG2) The matrices $\Gamma^{(n)}$, $\mathscr{A}^{(n)}$, and their dotted counterparts satisfy

$$|(\Gamma^{(n)1/2})_{uv} - \dot{\Gamma}^{(n)1/2})_{uv}| \leqslant \gamma_8 L^{-2n}(1+\mu_n^2)\, e^{-2\beta|u-v|}$$

$$|\mathscr{A}_{xu}^{(n)} - \dot{\mathscr{A}}_{xu}^{(n)}| \leqslant \gamma_9 L^{-n}(1+\mu_n^2)\, e^{-2\beta|x-u|}$$

where $\mu_n = L^n \mu_0$.

Then one can inductively show that

$$\begin{aligned}
&|\tilde{\tilde{V}}_{k,Y}^{(n)}(\Psi) - \dot{\tilde{\tilde{V}}}_{k,Y}^{(n)}(\dot{\Psi})| \\
&\quad \leqslant \gamma_{10}[(1+\mu_n^2)\,\varepsilon_n^{12-k}\,|\log \varepsilon_n|\, L^{-n/3} \\
&\qquad + \varepsilon_n^{13-k}\,|\log \varepsilon_n|\;|\Psi - \dot{\Psi}|_Y] \\
&\qquad \times \exp\left[\zeta_n \int_Y^{(n)} |\Psi_y - \dot{\Psi}_y|^2\, dy - \alpha \mathscr{L}(Y)\right], \qquad k = 2, 4, 6, 8 \quad (3.8)
\end{aligned}$$

for Ψ, $\dot{\Psi} \in 3\varepsilon_n^{-1}\mathscr{K}_n(Y)$ and for some $\alpha > 0$, where $\tilde{\tilde{V}}_{8,Y}^{(n)}$ and $\dot{\tilde{\tilde{V}}}_{8,Y}^{(n)}$ stand for the remainder terms ($\geqslant$8th-order parts) and $\varepsilon_n = \lambda_n^{1/4}$, $\zeta_n = 5/2(L^2/2)^n\,\varepsilon_n^2$, $|\Psi|_Y = \max_{y \in Y} |\Psi_y|$, and

$$\mathscr{K}_n(Y) = \{\Psi \in \mathbb{C}^Y |\ |\Psi|_Y \leqslant \theta_1 \text{ and } |\partial\Psi|_Y \leqslant \theta_2\}$$

The constants θ_1, $\theta_2 > 0$ are properly chosen. In the right-hand side of (3.8), the extra factor $\exp(\zeta_n \int_Y^{(n)} |\Psi_y - \dot{\Psi}_y|^2\, dy)$ is introduced when one improves the large-field bound by means of the "consistency condition" (see ref. 3). Furthermore, the proof of (3.8) needs a scaling of the fields Ψ and $\dot{\Psi}$. This procedure, however, causes a harmful enlargement of the constant ζ_n. The following lemma eliminates the difficulty by suppressing the dependence of the extra factor on the scaling.

Lemma 3.1. Let U and $\dot{U}$ be even analytic functions on $(1/2\varepsilon_n) L^{1/2}\mathscr{K}_{n+1}(X)$ such that $U(t\Psi)$ and $\dot{U}(t\Psi)$, $|t| \leqslant 1$, have zeros of the order $k > 0$ at $t = 0$. Then, the bound

$$|U(\Psi) - \dot{U}(\dot{\Psi})| \leqslant (\delta + \rho\, |\Psi - \dot{\Psi}|_X) \exp\left(2L^2\zeta_n \int_X^{(n+1)} |\Psi_x - \dot{\Psi}_x|^2\, dx\right)$$

$$\Psi, \dot{\Psi} \in \frac{1}{2\varepsilon_n} L^{1/2}\mathscr{K}_{n+1}(X)$$

implies

$$|U(\Psi) - \dot{U}(\dot{\Psi})| \leqslant 36L^{-1/2-k/4}(\delta + \rho L^{1/4}\, |\Psi - \dot{\Psi}|_X)$$

$$\times \exp\left(\zeta_{n+1} \int_X^{(n+1)} |\Psi_x - \dot{\Psi}_x|^2\, dx\right) \quad \Psi, \dot{\Psi} \in 3\varepsilon_{n+1}^{-1}\mathscr{K}_{n+1}(X)$$

Proof. The even function U has "the partial polarization"

$$U(\Psi, \Psi') = U((\Psi + \Psi')/2) - U((\Psi - \Psi')/2)$$

Defining $\dot{U}(\Psi, \Psi')$ similarly, consider the function $u(t, s) = U(ts\Psi, t\Psi) - \dot{U}(ts\Psi, t\dot{\Psi})$ and apply the maximum principle to $t^{-k}s^{-1}u(t, s)$. ∎

As for the large-field bound, we have

$$\begin{aligned}
&|g_X^{(n)D}(\Psi) - \dot{g}_X^{(n)D}(\dot{\Psi})| \\
&\leqslant \gamma_{11}[\varepsilon_n + (1 + \mu_n^2)\, L^{-n/3}\, |\Psi - \dot{\Psi}|_X] \\
&\quad \times \exp\left[\theta\, |D \cap X| - \frac{1}{2}\varepsilon_n^2 \int_{D\cap X}^{(n)} |\Psi_x|^2 \wedge |\dot{\Psi}_x|^2\, dx\right. \\
&\quad + 20\varepsilon_n^4 \int_{D\cap X}^{(n)} (\operatorname{Im} \Psi_x)^4 \vee (\operatorname{Im} \dot{\Psi}_x)^4\, dx \\
&\quad \left. + \zeta_n \int_X^{(n)} |\Psi_x - \dot{\Psi}_x|^2\, dx - \alpha\mathscr{L}(X)\right]
\end{aligned} \tag{3.9}$$

for $\Psi \in \varepsilon_n^{-1}\mathscr{D}_n(D, X)$ and $\dot{\Psi} \in \varepsilon_n^{-1}\dot{\mathscr{D}}_n(D, X)$, where $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and

$$\mathscr{D}_n(D, X) = \{\Psi \in \mathbb{C}^X |\text{ there exist } \phi \in \mathbb{R}^{\Lambda_n} \text{ and } \tilde{\Psi} \in \mathscr{K}_n(X) \text{ such that } \Psi = \mathscr{A}^{(n)}\phi + \tilde{\Psi} \text{ on } X \text{ and } D(\mathscr{A}^{(n)}\phi) \subset D\}$$

Here, $D(\Psi)$ denotes the smallest paved set D satisfying

$$|\Psi_x| \leqslant 2\theta_1 \exp[(\alpha/10)\, d(x, \sim D)], \qquad x \in X$$

where $\sim D = L^{-n}\Lambda_0 \backslash D$. Put $d(x, \phi) = \infty$. The $\dot{\mathscr{D}}_n(D, X)$ is defined similarly. In order to obtain (3.9), one first makes a straightforward bound for $\Psi' \in (1/6\varepsilon_n)\, L^{1/2}\mathscr{D}_n(D, X)$ and $\dot{\Psi}' \in (1/6\varepsilon_n)\, L^{1/2}\dot{\mathscr{D}}_n(D, X)$. Next, for these Ψ' and $\dot{\Psi}'$, one chooses the minimal paved sets D', $\dot{D}' \subset D$ such that $\Psi' \in (1/6\varepsilon_n)\, L^{1/2}\mathscr{D}_n(D', X)$ and $\dot{\Psi}' \in (1/6\varepsilon_n)\, L^{1/2}\dot{\mathscr{D}}', X)$ hold. Then, putting $D^* = D' \cup \dot{D}'$ and using the consistency condition which expresses $g_X^{(n)D'}$ in terms of $g_{X'}^{(n)D^*}$, $X' \subset X$, one obtains (3.9).

Remarks. 1. In the above arguments, the inductive step proceeds without assuming that $\tilde{\tilde{V}}^{(n)}$ and $\dot{\tilde{\tilde{V}}}^{(n)}$ were coming from the model defined by (3.1), (3.2). That is, one can add some quadratic terms to $\tilde{\tilde{V}}_2^{(n)}$ and $\dot{\tilde{\tilde{V}}}_2^{(n)}$. (However, an addition of higher-order terms will be troublesome because of the consistency condition.)

2. The effective potential $V^{(n)}$ was defined so that $V^{(n)}(0) = 0$. However, when one estimates the partition function, one has to take the zeroth-order term into account.

3. The implicit terms $\tilde{\tilde{V}}_{k,Y}^{(n+1)}$, $k = 2, 4, 6, 8$, and $g_Y^{(n+1)}$ depend on $\tilde{\tilde{V}}_{k,X}^{(n)}$, $k = 2, 4, 6, 8$, and $g_X^{(n)}$ such that $X \subset LY$ but not on ones such that $X \not\subset LY$. This is the case for the zeroth-order term.

3.2. Free Energy

Under the assumptions (G1)–(G3), one can show the following:

$$\gamma_{12}\mu_n^2 \leqslant (G^{(n)-1})_{uu} \leqslant \gamma_{13}\mu_n^2, \qquad u \in \Lambda_n \tag{3.10}$$

$$|(G^{(n)-1})_{uv}| \leqslant \gamma_{14}\mu_n e^{-\beta|u-v|}, \qquad u, v \in \Lambda_n, \quad \mu \neq v \tag{3.11}$$

for a sufficiently large μ_n (say $\geqslant \gamma_{15}$), where $\Lambda_n = L^{-n}\Lambda_0 \cap \mathbb{Z}^3$. For the proof, see Proposition A.1 in ref. 5.

Now choose an integer n_* such that $\gamma_{15} \leqslant \mu_{n_*}$ holds and ε_{n_*} is small (say $\leqslant \gamma_{16}$) so that the block spin machine works. (Such an integer n_* exists if λ_0/μ_0 is small.) Stop the iteration here.

According to ref. 5, one estimates the free energy for the n_*th effective theory:

$$\begin{aligned} f &= -\log Z \\ &= -\log \int d\mu_{G^{(n_*)}}(\phi) \exp[-V^{(n_*)}(\mathscr{A}^{(n_*)}\phi)] \end{aligned} \tag{3.12}$$

In what follows, let us omit superscript and subscript n_* and write

$$G^{(n_*)} = G, \qquad V^{(n_*)}(\mathscr{A}^{(n_*)}\phi) = V(\mathscr{A}\phi), \qquad \Lambda_{n_*} = \Lambda, \quad \text{etc.}$$

Introduce the matrices D and F defined by

$$D_{uv} = (G^{-1})_{uu}\delta_{uv}, \qquad u, v \in \Lambda$$
$$F = G^{-1} - D$$

and rewrite Z as

$$Z = Z_0^{-1} \sum_{p \in \mathbb{Z}_+^\Lambda} \int_{\varepsilon^{-1}F_{L^{-n_*}\Lambda_0}(p)} d\mu_{D^{-1}}(\phi) \exp[-V(\mathscr{A}\phi)]$$

where

$$F_X(p) = \{\phi \in \mathbb{R}^{X \cap \Lambda} \mid p_u \leqslant |\phi_u| < p_u + 1,\ u \in X \cap \Lambda\}, \qquad p = (p_u) \in \mathbb{Z}_+^{X \cap \Lambda}$$

and

$$Z_0 = \int d\mu_{D^{-1}}(\phi) \exp\left(-\frac{1}{2}\phi F \phi\right) \tag{3.13}$$

Proposition 3.2. For an integer n_* satisfying

$$\gamma_{15} \leqslant \mu_{n_*} \tag{3.14}$$
$$\lambda_{n_*} \leqslant \gamma_{16} \tag{3.15}$$
$$\mu_{n_*} \leqslant 2^{n_*/4} \tag{3.16}$$
$$\gamma_{17} \leqslant n_* \tag{3.17}$$

the free energies f and $\dot{f}$ defined by (3.12) (and its dotted version) have the expressions

$$f = -\sum_{Y \subset L^{-n_*}\Lambda_0} W_Y + \log Z_0$$
$$\dot{f} = -\sum_{Y \subset L^{-n_*}\Lambda_0} \dot{W}_Y + \log \dot{Z}_0$$

with the bounds

$$|W_Y|, |\dot{W}_Y| \leqslant \exp[-\tfrac{1}{5}\alpha\mathscr{L}(Y)] \tag{3.18}$$
$$|W_Y - \dot{W}_Y| \leqslant \gamma_{18}(1 + \mu_{n_*}^2) L^{-n_*/3} \exp[-\tfrac{1}{6}\alpha\mathscr{L}(Y)], \qquad Y \subset L^{-n_*}\Lambda_0 \tag{3.19}$$

where Z_0 and $\dot{Z}_0$ are defined by (3.13) and its dotted version, respectively.

Remarks. 1. W_Y ($\dot{W}_Y$) depends on

$$\tilde{V}_{k,X}^{(n_*)}, \quad k=2,4,6, \qquad \tilde{\tilde{V}}_{8,X}^{(n_*)}, \qquad g_X^{(n_*)D}$$

(on their dotted versions, respectively) such that $X \subset Y$ and not on those such that $X \not\subset Y$. On the other hand, Z_0 and $\dot{Z}_0$ are independent of all of them.

2. As an application of the proposition, one can estimate the correlations. Add an extra term $\sum h_u \phi_u$ (as in ref. 5) or $\sum h_{uv} \phi_u \phi_v$ (as in Section 4 of this paper) to the effective potential. Since correlations are derivatives of the free energy with respect to h, one can bound them by the maximum of $|f|$ on a certain complex region of h (see ref. 5 or this paper, Section 4). As a result, one finds that the present lattice model lies in the high-temperature region. At the same time, one obtains the ultraviolet stability bound for the Schwinger functions.

4. ROTATION INVARIANCE

In this section, I prove the rotation invariance of the Schwinger function S_{2l} as an application of Proposition 3.2. That is, I show the following equality:

$$\int_{(\mathbb{R}^3)^{2l}} dx_1\, dx_2 \cdots dx_{2l}\, S_{2l}(x_1, x_2,..., x_{2l}) \times \left[\prod_{k=1}^{2l} \omega_k(x_k) - \prod_{k=1}^{2l} \omega_k(\Theta^{-1} x_k) \right] = 0 \tag{4.1}$$

where $\omega_k \in \mathscr{S}(\mathbb{R}^3)$, $k = 1, 2,..., 2l$, and Θ denotes the rotation around the z axis by the angle $\sin^{-1}(3/5)$. The invariance with respect to any other rotation follows from (4.1), since $\pi^{-1} \sin^{-1}(3/5)$ is an irrational number.

4.1. Lattice Approximation

I prove (4.1) by showing

$$\lim_{N_1 \to \infty} \int^{(0)}_{(L^{-N_1}\mathbb{Z}^3)^{2l}} dx_1\, dx_2 \cdots dx_{2l}\, S_{2l}^{(N_1)}(x_1, x_2,..., x_{2l}) \times \left[\prod_{k=1}^{2l} \omega_k(x_k) - \prod_{k=1}^{2l} \omega_k(\Theta^{-1} x_k) \right] = 0 \tag{4.2}$$

and, if necessary, by taking a subsequence [see (2.9)]. For a while, fix $\omega_k \in \mathscr{S}(\mathbb{R}^3)$, $k = 1, 2,..., 2l$.

Let $[x]_{N_1}$ denote the point in $L^{-N_1}\mathbb{Z}^3$ nearest to $x \in \mathbb{R}^3$. Then, the mapping $x \mapsto [\Theta^{-1}x]_{N_1}$ turns out to be a one-to-one correspondence on $L^{-N_1}\mathbb{Z}^3$ and its inverse is given by $[\Theta x]_{N_1}$. Since ω_k, $k = 1, 2,..., 2l$, is uniformly continuous and the ultraviolet stability bound[2] yields

$$\int_{(L^{-N_1}\mathbb{Z}^3)^{2l-1}}^{(0)} dx_2 \cdots dx_{2l} \, |S_{2l}^{(N_1)}(x_1, x_2,..., x_{2l})| < \text{const} \tag{4.3}$$

for a constant independent of N_1 and x_1, (4.2) is equivalent to:

$$\begin{aligned} \lim_{N_1 \to \infty} \int_{(L^{-N_1}\mathbb{Z}^3)^{2l}}^{(0)} & dx_1 \, dx_2 \cdots dx_{2l} \prod_{k=1}^{2l} \omega_k(x_k) \\ & \times [S_{2l}^{(N_1)}(x_1, x_2,..., x_{2l}) \\ & - S_{2l}^{(N_1)}([\Theta x_1]_{N_1}, [\Theta x_2]_{N_1},..., [\Theta x_{2l}]_{N_1})] = 0 \end{aligned} \tag{4.4}$$

For $\nu = 0, 1, 2,...$, we denote by F_ν the family of all functions which have compact supports and are constant on each box

$$B_\nu(L^{-\nu}u) = \{x \in \mathbb{R}^3 | \; |x - L^{-\nu}u| < L^{-\nu}/2\}, \qquad u \in \mathbb{Z}^3$$

In order to show the equality (4.4) for $\omega_k \in \mathscr{S}(\mathbb{R}^3)$, it suffices to show it for $\omega_k \in \bigcup_\nu F_\nu$. Let $\omega_k \in F_\nu$, $k = 1, 2,..., 2l$, and put $D = L^\nu(\bigcup_k \operatorname{supp} u_k)$. Then, (4.4) is reduced to

$$\begin{aligned} \lim_{N_1 \to \infty} & \sup_{u_1, u_2,..., u_{2l} \in \Lambda_{N_1 - \nu} \cap D} \\ & \times \left| \int_{B_\nu(L^{-\nu}u_1)}^{(0)} dx_1 \int_{B_\nu(L^{-\nu}u_2)}^{(0)} dx_2 \cdots \int_{B_\nu(L^{-\nu}u_{2l})}^{(0)} dx_{2l} \right. \\ & \times \{S_{2l}^{(N_1)}(x_1, x_2,..., x_{2l}) \\ & \left. - S_{2l}^{(N_1)}([\Theta x_1]_{N_1}, [\Theta x_2]_{N_1},..., [\Theta x_{2l}]_{N_1})\} \right| = 0 \end{aligned} \tag{4.5}$$

Since (4.5) for an integer $\nu = \nu_0$ follows from that for $\nu = \nu_0 + 1$, one can assume that ν is larger than an arbitrarily fixed constant. Hereafter, ν and D will be fixed.

4.2. Rotated System

The function $S_{2l}^{(N_1)}([\Theta x_1]_{N_1}, [\Theta x_2]_{N_1},..., [\Theta x_{2l}]_{N_1})$ is regarded as a correlation of another lattice system: call it *the rotated system.* Let $[x]$ denote the point in $\mathbb{Z}^3$ nearest to $x \in \mathbb{R}^3$. Then, the mapping defined by

$u \in \mathbb{Z}^3 \mapsto [\Theta u] \in \mathbb{Z}^3$ has the inverse $u \mapsto [\Theta^{-1} u]$. For the infinite-volume limit $\bar{G}^{\mathrm{st}}$ of the standard Gaussian covariance G^{st} [see (2.5)], define the matrix $\bar{G}^{\mathrm{rot}}$ by

$$\bar{G}^{\mathrm{rot}}_{uv} = \bar{G}^{\mathrm{st}}_{[\Theta u][\Theta v]}, \qquad u, v \in \mathbb{Z}^3 \tag{4.6}$$

and denote its periodization on Λ_0 by G^{rot}:

$$G^{\mathrm{rot}}_{uv} = \sum_{w \in \mathbb{Z}^3} \bar{G}^{\mathrm{rot}}_{u + L^{N_2} w, v} \tag{4.7}$$

Then,

$$\begin{aligned} &S^{(N_1)}_{2l}([\Theta x_1]_{N_1}, [\Theta x_2]_{N_1}, \ldots, [\Theta x_{2l}]_{N_1}) \\ &\quad = \lim_{N_2 \to \infty} \dot{S}^{(N_1, N_2)}_{2l}(x_1, x_2, \ldots, x_{2l}) \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} &\dot{S}^{(N_1, N_2)}_{2l}(x_1, x_2, \ldots, x_{2l}) \\ &\quad = L^{N_1 l} \int d\mu_{G^{\mathrm{rot}}}(\phi) \exp[-\dot{V}^{(0)}(\phi)]\, \phi_{L^{N_1} x_1} \phi_{L^{N_1} x_2} \cdots \phi_{L^{N_1} x_{2l}} \end{aligned} \tag{4.9}$$

$$\begin{aligned} \dot{V}^{(0)}(\phi) = \lambda_0 \sum_{u \in \Lambda_0} \phi_u^4 - 6\lambda_0 \sum_{u \in \Lambda_0} G^{\mathrm{rot}}_{uu} \phi_u^2 \\ + 48\lambda_0^2 \sum_{u \in \Lambda_0} \sum_{v \in \Lambda_0} (G^{\mathrm{rot}}_{uv})^3 \phi_u^2 \end{aligned} \tag{4.10}$$

Thus, (4.5) is reduced to

$$\begin{aligned} &\lim_{N_1 \to \infty} \lim_{N_2 \to \infty} \sup_{u_1, u_2, \ldots, u_{2l} \in \Lambda_{N_1 - \nu} \cap D} \\ &\quad \times \left| \int^{(0)}_{B_\nu(L^{-\nu} u_1)} dx_1 \cdots \int^{(0)}_{B_\nu(L^{-\nu} u_{2l})} dx_{2l} \, [S^{(N_1, N_2)}_{2l}(x_1, x_2, \ldots, x_{2l}) \right. \\ &\quad \left. - \dot{S}^{(N_1, N_2)}_{2l}(x_1, x_2, \ldots, x_{2l})] \right| = 0 \end{aligned} \tag{4.11}$$

4.3. Mutual Difference

As is seen from (4.11) [and (2.4), (4.9)], our task is to compare two ϕ^4 systems on Λ_0 with distinct Gaussian measures $d\mu_{G^{\mathrm{st}}}$ and $d\mu_{G^{\mathrm{rot}}}$.

Proposition 4.1. If $m_0 \leqslant 1 \leqslant L^N m_0$, the Gaussian covariances $G^{(0)} = G^{\mathrm{st}}$ and $\dot{G}^{(0)} = G^{\mathrm{rot}}$ satisfy (G1)–(G3) (and their dotted versions) and

(ΔG1)–(ΔG2) for some positive constants $\beta, \gamma_1, \gamma_3, \gamma_6$, and γ_7 and for $\mu_0 = m_0/16$.

For the proof, see the Appendix.

Assuming Proposition 4.1 and putting $n = N_1 - v$, we can apply the result in Section 3 to the original and the rotated systems. Note that λ_n is small if v is sufficiently large. Consequently, we can write

$$\int_{B_v(L^{-v}u_1)}^{(0)} dx_1 \cdots \int_{B_v(L^{-v}u_{2l})}^{(0)} dx_{2l}\, S_{2l}^{(N_1)}(x_1, x_2, \ldots, x_{2l})$$
$$= (L^{vl}/Z_{N_1-v}) \int d\mu_{G^{(N_1-v)}}(\phi) \exp[-V^{(N_1-v)}(\mathscr{A}^{(N_1-v)}\phi)]\, \phi_{u_1} \cdots \phi_{u_{2l}} \tag{4.12}$$

and

$$\int_{B_v(L^{-v}u_1)}^{(0)} dx_1 \cdots \int_{B_v(L^{-v}u_{2l})}^{(0)} dx_{2l}\, \dot{S}_{2l}^{(N_1)}(x_1, x_2, \ldots, x_{2l})$$
$$= (L^{vl}/\dot{Z}_{N_1-v}) \int d\mu_{\dot{G}^{(N_1-v)}}(\phi) \exp[-\dot{V}^{(N_1-v)}(\dot{\mathscr{A}}^{(N_1-v)}\phi)]\, \phi_{u_1} \cdots \phi_{u_{2l}} \tag{4.13}$$

where Z_{N_1-v} and $\dot{Z}_{N_1-v}$ are normalization constants.

In order to calculate the right-hand sides of (4.12) and (4.13), add

$$\sum_{u,v \in \Lambda_{N_1-v}} h_{uv} \phi_u \phi_v$$

to the effective potentials (see Section 3.1, Remark 1). We have to write the addendum in terms of the variable Ψ: noting that

$$\phi_u = \int_{B_0(u)}^{(N_1-v)} (\mathscr{A}^{(N_1-v)}\phi)_x\, dx = \int_{B_0(u)}^{(N_1-v)} (\dot{\mathscr{A}}^{(N_1-v)}\phi)_x\, dx$$

put

$$V^{(N_1-v)}(h; \Psi)$$
$$= V^{(N_1-v)}(\Psi) + \sum_{u,v \in \Lambda_{N_1-v}} h_{uv} \int_{B_0(u)}^{(N_1-v)} dx \int_{B_0(u)}^{(N_1-v)} dy\, \Psi_x \Psi_y \tag{4.14}$$

and

$$\dot{V}^{(N_1-v)}(h; \dot{\Psi})$$
$$= \dot{V}^{(N_1-v)}(\dot{\Psi}) + \sum_{u,v \in \Lambda_{N_1-v}} h_{uv} \int_{B_0(u)}^{(N_1-v)} dx \int_{B_0(v)}^{(N_1-v)} dy\, \dot{\Psi}_x \dot{\Psi}_y \tag{4.15}$$

where the complex variables $h_{uv} = h_{vu}$, $u, v \in \Lambda_{N_1 - v}$, are assumed to obey the bound

$$|h_{uv}| \leqslant \varepsilon_{N_1 - v}^{12} e^{-\alpha|u-v|} = (L^{-v}\lambda_*)^3 e^{-\alpha|u-v|} \tag{4.16}$$

(More precisely, add the term to the quadratic parts $V_2^{(N_1-v)}$ and $\dot{V}_2^{(N_1-v)}$.) Note that the right-hand sides of (4.12) and (4.13) can be written in terms of the derivatives (at $h_{uv} = 0$) of

$$\log Z_{N_1, N_2, v}(h) = \int d\mu_{G^{(N_1-v)}}(\phi) \exp[-V^{(N_1-v)}(h; \mathscr{A}^{(N_1-v)}\phi)] \tag{4.17}$$

and of

$$\log \dot{Z}_{N_1, N_2, v}(h) = \int d\mu_{\dot{G}^{(N_1-v)}}(\phi) \exp[-\dot{V}^{(N_1-v)}(h; \dot{\mathscr{A}}^{(N_1-v)}\phi)] \tag{4.18}$$

respectively. Then, in order to obtain (4.11), we have to show

$$\lim_{N_1\to\infty} \lim_{N_2\to\infty} \sup_{u_1, u_2, \ldots, u_{2l} \in \Lambda_{N_1-v} \cap D} \times \left| \frac{\partial}{\partial h_{u_1 u_2}} \cdots \frac{\partial}{\partial h_{u_{2l-1} u_{2l}}} \log \frac{Z_{N_1, N_2, v}(h)}{\dot{Z}_{N_1, N_2, v}(h)} \bigg|_{h=0} \right| = 0 \tag{4.19}$$

I now resume the block spin transformation. Taking the constant terms $[V^{(n)}(h; 0)$ and $\dot{V}^{(n)}(h; 0)]$ into account, trace the trajectories $V^{(n)}(h; \Psi)$ and $\dot{V}^{(n)}(h; \Psi)$ for $N_1 - v \leqslant n \leqslant n_*$, where n_* is the smallest integer satisfying (3.14)–(3.17). In terms of $v' = n_* - N_1$, the conditions (3.14)–(3.17) are rewritten as

$$\gamma_{15} \leqslant L^{v'} m_*/16$$

$$L^{v'}\lambda_* \leqslant \gamma_{16}$$

$$(2^{-1/4}L)^{v'} m_*/16 \leqslant 2^{N_1/4}$$

$$\gamma_{17} \leqslant N_1 + v'$$

since $\mu_{n_*} = L^{v'} m_*/16$ and $\lambda_{n_*} = L^{v'}\lambda_*$. Under the assumption $\lambda_*/m_* \leqslant \gamma_{16}(16L\gamma_{15})^{-1}$, the integers

$$v' = [\log(16\gamma_{15}/m_*)/\log L] + 1$$

and

$$N_1 \geqslant \gamma_{17} \vee \left[\frac{\log(L^4/2)}{\log L} \log(16\gamma_{15}) + \frac{\log 2}{\log L} \log m_* \right]$$

satisfy the above conditions.

Thus, Proposition 3.2 implies

$$\log[Z_{N_1,N_2,v}(h)/\dot{Z}_{N_1,N_2,v}(h)] = \sum_{Y \subset L^{-n_*}\Lambda_0} [W_Y(h) - \dot{W}_Y(h)] - \log(Z_0/\dot{Z}_0) \tag{4.20}$$

with the bounds (3.18) and (3.19). Note that Z_0 and $\dot{Z}_0$ are independent of h_{uv}'s and that $W_Y(h)$ depends only on h_{uv}'s such that $L^{-(v+v')}u$, $L^{-(v+v')}v \in Y$ (see Section 3.1, Remark 3; and Section 3.2, Remark 1). Therefore, it holds that

$$\frac{\partial}{\partial h_{u_1u_2}} \cdots \frac{\partial}{\partial h_{u_{2l-1}u_{2l}}} \log \frac{Z_{N_1,N_2,v}(h)}{\dot{Z}_{N_1,N_2,v}(h)} = \sum_{Y \subset L^{-n_*}\Lambda_0} \chi(L^{-(v+v')}u_j \in Y, j=1, 2,..., 2l) \times \frac{\partial}{\partial h_{u_1u_2}} \cdots \frac{\partial}{\partial h_{u_{2l-1}u_{2l}}} [W_Y(h) - \dot{W}_Y(h)]$$

where χ is the characteristic function. The above equality combined with the bound (3.19) yields

$$\left| \frac{\partial}{\partial h_{u_1u_2}} \cdots \frac{\partial}{\partial h_{u_{2l-1}u_{2l}}} \log \frac{Z_{N_1,N_2,v}(h)}{\dot{Z}_{N_1,N_2,v}(h)} \right|_{h=0} \Bigg| \leqslant \gamma_{18} \left[1 + \left(\frac{L^{v'} m_*}{16} \right)^2 \right] L^{3lv} \lambda_*^{-3l} L^{-(N_1+v')l/3} e^{l\alpha \operatorname{diam}(D)}$$

where $\operatorname{diam}(D) = \max_{x,y \in D} |x-y|$ and I have used the Cauchy estimate. This implies (4.19).

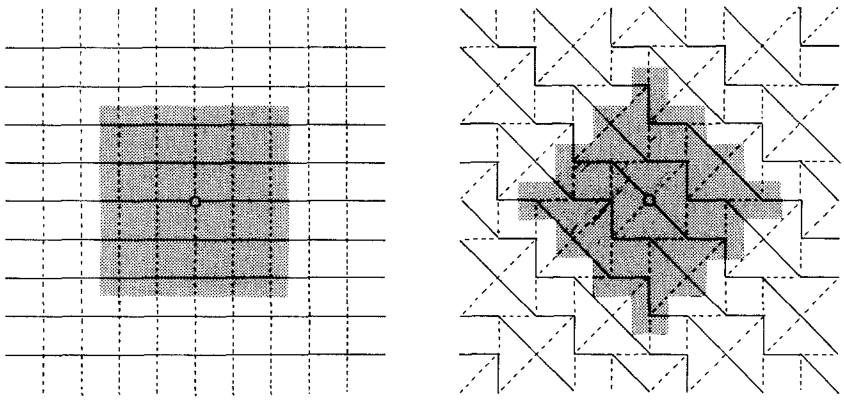
Thus, I have proved the theorem assuming Proposition 4.1.

APPENDIX

I prove Proposition 4.1. Let $\bar{G}^{(0)}$ and $\dot{\bar{G}}^{(0)}$ denote the infinite-volume limits of $G^{(0)} = G^{\text{st}}$ and $\dot{G}^{(0)} = G^{\text{rot}}$, respectively. For $G^{(0)}$, (G1) and (G2) can be shown similarly as in refs. 4 and 9. (G3) is trivial.

A1. Bounds on Rotated Gaussian Kernels

Figure 1 shows interacting bonds in $\mathbb{Z}^3$ for the rotated Gaussian measure $d\mu_{\dot{G}^{(0)}}$. Note that $\dot{G}^{(0)}$ is invariant with respect to translations by vectors $\equiv 0 \pmod 5$, while $\dot{G}^{(n)}$, $n \geqslant 1$, has the full translation invariance. In



(a) (b)

Fig. 1. The Gaussian interactions (bonds) in the xy plane are illustrated for (a) G^{st} and (b) G^{rot}.

what follows, I shall deal with the kernels in the infinite-volume limit. Finite-volume versions are obtained as their periodizations.

One can show the Ornstein–Zernike decay of $\dot{\bar{G}}^{(0)}$ from that of $\bar{G}^{(0)}$ by putting $\mu_0 = m_0/16$ and replacing γ_1 properly.

Let us show

$$|(\dot{\bar{G}}^{(n)-1})_{uv}| \leqslant \text{const} \cdot (1 + L^{2n} m_0^2)\, e^{-3\beta|u-v|} \tag{A.1}$$

where const is a numerical constant. Assume $n \geqslant 1$ and put $L = 5$ without loss of generality. Consider the Fourier transforms of $\bar{G}^{(n)}$ and $\dot{\bar{G}}^{(n)}$:

$$\hat{G}_n(p) = \sum_{u \in \mathbb{Z}^3} e^{-ipu} \bar{G}^{(n)}_{u0}, \qquad p \in \mathbb{R}^3/2\pi\mathbb{Z}^3$$

$$\dot{\hat{G}}_n(p) = \sum_{u \in \mathbb{Z}^3} e^{-ipu} \dot{\bar{G}}^{(n)}_{u0}, \qquad p \in \mathbb{R}^3/2\pi\mathbb{Z}^3$$

As is easily seen, it holds that

$$\dot{\hat{G}}_n(p) = \sum_{r \in R_n} 5^{-2n} \hat{G}_0(\Theta(p + 2\pi r)/5^n)\, |\dot{X}_n(\Theta(p + 2\pi r)/5^n)|^2 \tag{A.2}$$

where

$$\dot{X}_n(q) = 5^{-3n} \sum_{|u| \leqslant (5^n - 1)/2} e^{-iq[\Theta u]}, \qquad q \in \mathbb{R}^3/2\pi\mathbb{Z}^3 \tag{A.3}$$

and $R_n = \{(5^{n-1}s_1 + r_1', 5^{n-1}s_2 + r_2', r_3) \mid r_1', r_2', r_3, s_1, s_2$ are integers such that $|r_i'| \leqslant (5^{n-1}-1)/2$, $i=1, 2$, $|r_3| \leqslant (5^n-1)/2$, and $|s_1|+|s_2| \leqslant 3\}$. By means of (A.2) and the decomposition

$$\hat{\dot{G}}_n(p) = \frac{\hat{\dot{G}}_n(p)}{5^{-2n}\hat{G}_0(5^{-n}\Theta p)} \cdot 5^{-2n}\hat{G}_0(5^{-n}\Theta p) \tag{A.4}$$

one can derive the lower bound for $|\hat{\dot{G}}_n(p)|$:

$$|\hat{\dot{G}}_n(p)| \geqslant \text{const} \cdot (1+5^{2n}m_0^2)^{-1}$$

for p in a complex neighborhood of $[-\pi, \pi]^3$ independent of n (see ref. 9). This implies (A.1).

The bound

$$|\dot{\Gamma}_{uv}^{(n)}| \leqslant \text{const} \cdot e^{-3\beta|u-v|}$$

is obtained similarly as in ref. 9. Note that we have to deal with the case $n=0$ separately (since $\dot{G}^{(0)}$ does not have translation invariance) and that we cannot assume $L=5$. In case $n=0$, the boundedness of $\hat{G}_0(\Theta p/L)/\hat{\dot{G}}_1(p)$ yields the necessary estimate. For the bound of $\dot{\Gamma}_{uv}^{(n)1/2}$, see ref. 9.

In order to estimate $\dot{\mathscr{A}}^{(n)}$, we have to bound $L^{1/2}(\dot{G}^{(0)}C^*\dot{G}^{(1)-1})_{Lx,u}$ and $L^{(n-1)/2}(\dot{G}^{(1)}C^{*n-1}\dot{G}^{(n)-1})_{L^{n-1}x,u}$.

A2. Mutual Differences

The infinite-volume version of (ΔG1) is

$$|L^n\bar{G}_{L^nx,L^ny}^{(0)} - L^n\dot{\bar{G}}_{L^nx,L^ny}^{(0)}| \leqslant \text{const} \cdot L^{-n}\,|x-y|^{-2}$$
$$x \neq y, \qquad x, y \in L^{-n}\mathbb{Z}^3 \tag{A.5}$$

In order to prove (A.5), it suffices to show

$$|L^n\bar{G}_{L^nx,0}^{(0)} - (4\pi\langle x\rangle)^{-1}\exp(-L^n m_0\langle x\rangle)| \leqslant \text{const} \cdot L^{-n}\,|x|^{-2} \tag{A.6}$$

where $\langle x\rangle^2 = x_1^2+x_2^2+x_3^2$. Starting with the expression

$$L^n\bar{G}_{L^nx,0}^{(0)} = \int_{|q| \leqslant L^n\pi} \frac{dq}{(2\pi)^3}\, e^{iqx} L^{-2n}\left[2\sum_{j=1}^{3}\left(1-\cos\frac{q_j}{L^n}\right)+m_0^2\right]^{-1}$$

replace (i) the Fourier kernel by $(\langle q\rangle^2 + L^{2n}m_0^2)^{-1}$; (ii) the domain of integration $|q| \leqslant L^n\pi$ by $\langle q\rangle \leqslant L^n\pi$; and (iii) the latter by $\mathbb{R}^3$. Then one obtains the Fourier expansion of $(4\pi\langle x\rangle)^{-1}\exp(-L^n m_0\langle x\rangle)$. In each step, the difference can be estimated by virtue of integrations by parts.

I next show

$$|(\bar{G}^{(n)-1})_{uv} - (\dot{\bar{G}}^{(n)-1})_{uv}| \leqslant \text{const} \cdot (1 + L^{2n} m_0^2)\, e^{-3\beta|u-v|} \qquad \text{(A.7)}$$

It is needed to bound $\hat{G}_n(p)^{-1} - \dot{\hat{G}}_n(p)^{-1}$ in a complex neighborhood of $[-\pi, \pi]^3$. Put $L = 5$ and write

$$\hat{G}_n(p)^{-1} - \dot{\hat{G}}_n(p)^{-1} = Z_n(p)/Y_n(p) - \dot{Z}_n(p)/\dot{Y}_n(p)$$

where

$$Y_n(p) = 5^{2n} \hat{G}_n(p)/\hat{G}_0(p/5^n)$$
$$\dot{Y}_n(p) = 5^{2n} \dot{\hat{G}}_n(p)/\hat{G}_0(\Theta p/5^n)$$
$$Z_n(p) = 5^{2n}/\hat{G}_0(p/5^n)$$
$$\dot{Z}_n(p) = 5^{2n}/\hat{G}_0(\Theta p/5^n)$$

In order to bound $Y_n(p) - \dot{Y}_n(p)$, use (A.2) and the analogous expression for $\dot{\hat{G}}_n(p)$ together with the equality

$$\frac{\hat{G}_0(\Theta(p + 2\pi r)/5^n)}{\hat{G}_0(\Theta p/5^n)} - \frac{\hat{G}_0((p + 2\pi r)/5^n)}{\hat{G}_0(p/5^n)} = \int_0^{\sin^{-1}(3/5)} d\theta\, \frac{\dot{\hat{G}}_0(\Theta(\theta)(p + 2\pi r)/5^n)}{\hat{G}_0(\Theta(\theta)\, p/5^n)}$$

where $\Theta(\theta)$ denotes the rotation around the z axis by the angle θ. The right-hand side is easily estimated.

Since (A.7) yields a bound on $\Gamma^{(n)-1} - \dot{\Gamma}^{(n)-1}$, we can estimate $\Gamma^{(n)-1/2} - \dot{\Gamma}^{(n)-1/2}$ and hence $\Gamma^{(n)1/2} - \dot{\Gamma}^{(n)1/2}$.

Lastly, $\mathscr{A}^{(n)} - \dot{\mathscr{A}}^{(n)}$ is estimated similarly as $\dot{\mathscr{A}}^{(n)}$: $L^{1/2}(G^{(0)}C^*G^{(1)-1})_{L^n x, L^{n-1} y}$ and its dotted version are close to each other since both are close to δ_{xy}. The bound on the difference between $L^{(n-1)/2}(G^{(1)}C^{*n-1}G^{(n)-1})_{L^{n-1}y, u}$ and its dotted version is obtained in a straightforward way.

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